JP Journal of Fixed Point Theory and Applications

http://www.pphmj.com
http://dx.doi.org/10.17654/FP015020081
Volume 15, Number 2, 2020, Pages 81-91 **ISSN: 0973-4228**

NUMBER OF ZEROS OF POLAR DERIVATIVES OF POLYNOMIALS

P. Ramulu and G. L. Reddy

Department of Mathematics
M. V. S. Govt. Arts and Science College (Autonomous)
Mahabubnagar, Telangana, 509001, India
e-mail: ramulu.purra@gmail.com

School of Mathematics and Statistics
University of Hyderabad
Telangana, 500046, India
e-mail: glrsm@uohyd.ernet.in

Abstract

In this paper, we estimate the maximum number of zeros of polar derivatives of polynomials by considering more general coefficient conditions in a prescribed region. The results which we obtain generalize and improve some of the well known results.

1. Introduction

Let $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ denote the polar derivative of a polynomial $P(z)$ of degree n with respect to real or complex number α. Then polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the

Received: April 7, 2020; Accepted: June 29, 2020
2010 Mathematics Subject Classification: 30C10, 30C15.
Keywords and phrases: polar derivatives, zeros of polynomials, prescribed region.

ordinary derivative in the sense that $\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$. Many results on the location of zeros of polynomials and zeros of polar derivatives are available in the literature [1-5]. Concerning the number of zeros of the polynomial in the region $|z| \leq \frac{1}{2}$, the following result is due to Mohammad [6].

Theorem A. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that* $0 < a_0 \leq a_1 \leq \cdots \leq a_{n-1} \leq a_n$. *Then the number of zeros of* $P(z)$ *in* $|z| \leq \frac{1}{2}$, *does not exceed* $1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}$.

In this paper, we prove the following results.

Theorem 1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with* $Re(a_i) = \alpha_i$, $Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, \ldots, n$ *and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $\alpha a_1 + a_0 \neq 0$ *and*

$$[i+2]\alpha\alpha_{i+2} + [n-(i+1)]\alpha_{i+1} \geq (i+1)\alpha\alpha_{i+1} + (n-i)\alpha_i,$$

$$[i+2]\alpha\beta_{i+2} + [n-(i+1)]\beta_{i+1} \geq (i+1)\alpha\beta_{i+1} + (n-i)\beta_i,$$

for $i = 0, 1, 2, \ldots, n-2$. *Then the number of zeros of* $D_\alpha P(z)$ *in* $|z| \leq \frac{R}{C}$ $(C > 1,\ R > 0)$, *does not exceed*

$$\frac{1}{\log C} \log \frac{R^n[|n\alpha a_n + a_{n-1}| + n\alpha[\alpha_n + \beta_n] + [\alpha_{n-1} + \beta_{n-1}] - \alpha[\alpha_1 + \beta_1] - n[\alpha_0 + \beta_0] + |\alpha a_1 + na_0|]}{|\alpha a_1 + na_0|}, \quad \text{if } R \geq 1$$

and

$$\frac{1}{\log C} \log \frac{R[|n\alpha a_n + a_{n-1}| + n\alpha[\alpha_n + \beta_n] + [\alpha_{n-1} + \beta_{n-1}] - \alpha[\alpha_1 + \beta_1] - n[\alpha_0 + \beta_0] + |\alpha a_1 + na_0|]}{|\alpha a_1 + na_0|} \text{ if } R \leq 1.$$

Corollary 1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with* $Re(a_i) = \alpha_i$, $Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, \ldots, n$ *and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $\alpha a_1 + a_0 \neq 0$ *and*

$$[i+2]\alpha\alpha_{i+2} + [n-(i+1)]\alpha_{i+1} \geq (i+1)\alpha\alpha_{i+1} + (n-i)\alpha_i,$$

for $i = 0, 1, 2, \ldots, n-2$. *Then the number of zeros of* $D_\alpha P(z)$ *in* $|z| \leq r$, $0 < r < 1$, *does not exceed*

$$\frac{1}{\log\frac{1}{r}} \log \frac{\left[|n\alpha\alpha_n + \alpha_{n-1}| + n\alpha\alpha_n + \alpha_{n-1} - \alpha\alpha_1 - n\alpha_0 + |\alpha\alpha_1 + \alpha_0| + 2\sum_{i=0}^{n-1} |(i+1)\alpha\beta_{i+1} + (n-i)\beta_i|\right]}{|\alpha a_1 + na_0|}.$$

Corollary 2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with real coefficients and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $\alpha a_1 + a_0 \neq 0$ *and* $[i+2]\alpha a_{i+2} + [n-(i+1)]a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i$, *for* $i = 0, 1, 2, \ldots, n-2$. *Then the number of zeros of* $D_\alpha P(z)$ *in* $|z| \leq \frac{1}{2}$, *does not exceed*

$$\frac{1}{\log 2} \log \frac{[|n\alpha a_n + a_{n-1}| + n\alpha a_n + a_{n-1} - \alpha a_1 - a_0 + |\alpha a_1 + na_0|]}{|\alpha a_1 + na_0|}.$$

Remark 1. Taking $R = 1$, $C = \frac{1}{r}$, $0 < r < 1$, removing conditions on β_i in Theorem 1 and rearranging coefficients, we get Corollary 1.

Remark 2. Taking $R = 1$, $C = \frac{1}{2}$ and $\beta_i = 0$ for $i = 0, 1, \ldots, n$ in Theorem 1, we get Corollary 2.

Theorem 2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with* $Re(a_i) = \alpha_i$, $Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, \ldots, n$ *and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $\alpha a_1 + a_0 \neq 0$ *and*

$$[i+2]\alpha\alpha_{i+2} + [n-(i+1)]\alpha_{i+1} \leq (i+1)\alpha\alpha_{i+1} + (n-i)\alpha_i,$$

$$[i+2]\alpha\beta_{i+2} + [n-(i+1)]\beta_{i+1} \leq (i+1)\alpha\beta_{i+1} + (n-i)\beta_i,$$

for $i = 0, 1, 2, \ldots, n-2$. *Then the number of zeros of* $D_\alpha P(z)$ *in* $|z| \leq \frac{R}{C}$ $(C > 1,\ R > 0)$, *does not exceed*

$$\frac{1}{\log C}\log\frac{\begin{array}{c}R^n[|n\alpha a_n + a_{n-1}| - n\alpha[\alpha_n + \beta_n] - [\alpha_{n-1} + \beta_{n-1}]\\ + \alpha[\alpha_1 + \beta_1] + n[\alpha_0 + \beta_0] + |\alpha a_1 + na_0|]\end{array}}{|\alpha a_1 + na_0|}, \quad \textit{if } R \geq 1$$

and

$$\frac{1}{\log C}\log\frac{\begin{array}{c}R[|n\alpha a_n + a_{n-1}| - n\alpha[\alpha_n - \beta_n] + [\alpha_{n-1} + \beta_{n-1}]\\ + \alpha[\alpha_1 + \beta_1] + n[\alpha_0 + \beta_0] + |\alpha a_1 + na_0|]\end{array}}{|\alpha a_1 + na_0|}, \quad \textit{if } R \leq 1.$$

Corollary 3. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with* $Re(a_i) = \alpha_i$, $Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, \ldots, n$ *and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $\alpha a_1 + a_0 \neq 0$ *and*

$$[i+2]\alpha\alpha_{i+2} + [n-(i+1)]\alpha_{i+1} \leq (i+1)\alpha\alpha_{i+1} + (n-i)\alpha_i,$$

for some $i = 0, 1, 2, \ldots, n-2$. *Then the number of zeros of* $D_\alpha P(z)$ *in* $|z| \leq \frac{1}{2}$, *does not exceed*

$$\frac{1}{\log 2}\log\frac{\left[|n\alpha\alpha_n+\alpha_{n-1}|-n\alpha\alpha_n+\alpha_{n-1}-\alpha\alpha_1+n\alpha_0+|\alpha\alpha_1+\alpha_0|+2\sum_{i=0}^{n-1}|(i+1)\alpha\beta_{i+1}+(n-i)\beta_i|\right]}{|\alpha a_1+na_0|}.$$

Corollary 4. *Let* $P(z)=\sum_{i=0}^{n}a_iz^i$ *be a polynomial of degree n with real coefficients and* $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $\alpha a_1+a_0\neq 0$ *and* $[i+2]\alpha a_{i+2}+[n-(i+1)]a_{i+1}\leq(i+1)\alpha a_{i+1}+(n-i)a_i$, *for* $i=0,$ $1, 2, \ldots, n-2$. *Then the number of zeros of* $D_\alpha P(z)$ *in* $|z|\leq r$, $0<r<1$ *does not exceed*

$$\frac{1}{\log\frac{1}{r}}\log\frac{[|n\alpha a_n+a_{n-1}|-n\alpha\alpha_n-\alpha_{n-1}+\alpha\alpha_1+n\alpha_0+|\alpha a_1+na_0|]}{|\alpha a_1+na_0|}.$$

Remark 3. Taking $R=1$, $C=\frac{1}{2}$, and removing conditions on β_i in Theorem 2, we get Corollary 3.

Remark 4. Taking $R=1$, $C=\frac{1}{r}$, $0<r<1$ and $\beta_i=0$ for $i=0,$ $1, \ldots, n$ in Theorem 2, and by rearranging coefficients, we get Corollary 4.

By rearranging the coefficient in Theorems 1 and 2, we get the following Theorems 3 and 4.

Theorem 3. *Let* $P(z)=\sum_{i=0}^{n}a_iz^i$ *be a polynomial of degree n with* $Re(a_i)=\alpha_i$, $Im(a_i)=\beta_i$ *for* $i=0, 1, 2, \ldots, n$ *and* $D_\alpha P(z)=nP(z)+$ $(\alpha-z)P'(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $\alpha a_1+a_0\neq 0$ *and*

$$[i+2]\alpha\alpha_{i+2}+[n-(i+1)]\alpha_{i+1}\geq(i+1)\alpha\alpha_{i+1}+(n-i)\alpha_i,$$

$$[i+2]\alpha\beta_{i+2}+[n-(i+1)]\beta_{i+1}\leq(i+1)\alpha\beta_{i+1}+(n-i)\beta_i,$$

for $i = 0, 1, 2, \ldots, n-2.$ *Then the number of zeros of* $D_\alpha P(z)$ *in* $|z| \le \frac{R}{C}$ $(C > 1,\ R > 0),$ *does not exceed*

$$\frac{1}{\log C}\log\frac{\begin{array}{c}R^n[|n\alpha a_n + a_{n-1}| + n\alpha[\alpha_n - \beta_n] + [\alpha_{n-1} - \beta_{n-1}]\\ -\alpha[\alpha_1 - \beta_1] - n[\alpha_0 - \beta_0] + |\alpha a_1 + a_0|]\end{array}}{|\alpha a_1 + na_0|}, \quad \text{if } R \ge 1$$

and

$$\frac{1}{\log C}\log\frac{\begin{array}{c}R[|n\alpha a_n + a_{n-1}| + n\alpha[\alpha_n - \beta_n] + [\alpha_{n-1} - \beta_{n-1}]\\ -\alpha[\alpha_1 - \beta_1] - n[\alpha_0 - \beta_0] + |\alpha a_1 + a_0|]\end{array}}{|\alpha a_1 + na_0|}, \quad \text{if } R \le 1.$$

Theorem 4. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with* $Re(a_i) = \alpha_i,$ $Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, \ldots, n$ *and* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *be the polar derivative of* $P(z)$ *with respect to a real number* α *such that* $\alpha a_1 + a_0 \ne 0$ *and*

$$[i+2]\alpha\alpha_{i+2} + [n-(i+1)]\alpha_{i+1} \le (i+1)\alpha\alpha_{i+1} + (n-i)\alpha_i,$$

$$[i+2]\alpha\beta_{i+2} + [n-(i+1)]\beta_{i+1} \ge (i+1)\alpha\beta_{i+1} + (n-i)\beta_i,$$

for $i = 0, 1, 2, \ldots, n-2.$ *Then the number of zeros of* $D_\alpha P(z)$ *in* $|z| \le r,\ 0 < r < 1$ *does not exceed*

$$\frac{1}{\log\frac{1}{r}}\log\frac{\begin{array}{c}[|n\alpha a_n + a_{n-1}| - n\alpha[\alpha_n - \beta_n] - [\alpha_{n-1} - \beta_{n-1}]\\ +\alpha[\alpha_1 - \beta_1] + n[\alpha_0 - \beta_0] + |\alpha a_1 + na_0|]\end{array}}{|\alpha a_1 + na_0|}.$$

We need the following lemma for the proofs of the above theorems.

2. Lemma

Lemma 1 [7]**.** *If* $f(z)$ *is regular,* $f(0) \neq 0$ *and* $|f(z)| \leq M(R)$ *in* $|z| \leq 1$. *Then the number of zeros of* $f(z)$ *in* $|z| \leq \frac{R}{C}$, $(C > 1,\ R > 0)$ *does not exceed* $\frac{1}{\log C} \log \frac{M(R)}{|f(0)|}$.

3. Proof of the Theorems

Proof of Theorem 1. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with $Re(a_i) = \alpha_i$, $Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, \ldots, n$.

Denote by $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ to be the polar derivative of $P(z)$ with respect to the real number α of degree $n - 1$. This implies

$$D_\alpha P(z) = [n\alpha a_n + a_{n-1}]z^{n-1} + [(n-1)\alpha a_{n-1} + 2a_{n-2}]z^{n-2}$$
$$+ [(n-2)\alpha a_{n-2} + 3a_{n-3}]z^{n-3} + \cdots + [3\alpha a_3 + (n-2)a_2]z^2$$
$$+ [2\alpha a_2 + (n-1)a_1]z + [\alpha a_1 + na_0].$$

Now consider the polynomial $Q(z) = (1 - z)D_\alpha P(z)$ so that

$$Q(z) = -[n\alpha a_n + a_{n-1}]z^n + [n\alpha a_n + \{1 - (n-1)\alpha\}a_{n-1} - 2a_{n-2}]z^{n-1}$$
$$+ [(n-1)\alpha a_{n-1} + \{2 - (n-2)\alpha\}a_{n-2} - 3a_{n-3}]z^{n-2}$$
$$+ \cdots + [3\alpha a_3 + \{(n-2) - 2\alpha\}a_2 - (n-1)a_1]z^2$$
$$+ [2\alpha a_2 + \{(n-1) - \alpha\}a_1 - na_0]z + [\alpha a_1 + na_0]$$
$$= -[n\alpha a_n + a_{n-1}]z^n$$
$$+ \sum_{i=0}^{n-2} [[i+2]\alpha a_{i+2} + ([n - (i+1)] - [i+1]\alpha)a_{i+1} - (n-i)a_i]z^i$$
$$+ [\alpha a_1 + na_0].$$

For $|z| \le R,$ we have

$$
\begin{aligned}
|Q(z)| &\le |n\alpha a_n + a_{n-1}|R^n \\
&\quad + \sum_{i=0}^{n-2} |[[i+2]\alpha a_{i+2} + ([n-(i+1)] - [i+1]\alpha) a_{i+1} - (n-i) a_i]| R^i \\
&\quad + |\alpha a_1 + n a_0| \\
&\le |n\alpha a_n + a_{n-1}|R^n \\
&\quad + \sum_{i=0}^{n-2} |[[i+2]\alpha \alpha_{i+2} + ([n-(i+1)] - [i+1]\alpha) \alpha_{i+1} - (n-i) \alpha_i]| R^i \\
&\quad + \sum_{i=0}^{n-2} |[[i+2]\alpha \beta_{i+2} + ([n-(i+1)] - [i+1]\alpha) \beta_{i+1} - (n-i) \beta_i]| R^i \\
&\quad + |\alpha a_1 + n a_0| \\
&\le |n\alpha a_n + a_{n-1}|R^n \\
&\quad + \sum_{i=0}^{n-2} [[i+2]\alpha \alpha_{i+2} + ([n-(i+1)] - [i+1]\alpha) \alpha_{i+1} - (n-i) \alpha_i] R^i \\
&\quad + \sum_{i=0}^{n-2} [[i+2]\alpha \beta_{i+2} + ([n-(i+1)] - [i+1]\alpha) \beta_{i+1} - (n-i) \beta_i] R^i \\
&\quad + |\alpha a_1 + n a_0| \\
&\le \begin{cases} R^n[|n\alpha a_n + a_{n-1}| + n\alpha[\alpha_n + \beta_n] + [\alpha_{n-1} + \beta_{n-1}] \\ \quad - \alpha[\alpha_1 + \beta_1] - n[\alpha_0 + \beta_0] + |\alpha a_1 + n a_0|], \quad \text{if } R \ge 1, \\ R[|n\alpha a_n + a_{n-1}| + n\alpha[\alpha_n + \beta_n] + [\alpha_{n-1} + \beta_{n-1}] \\ \quad - \alpha[\alpha_1 + \beta_1] - n[\alpha_0 + \beta_0]] + |\alpha a_1 + n a_0|, \quad \text{if } R \le 1. \end{cases} \qquad (1)
\end{aligned}
$$

Applying Lemma 1 to $Q(z)$, we get that the number of zeros of $Q(z)$ in $|z| \leq \frac{R}{C}$ $(C > 1, R > 0)$, does not exceed

$$\text{(i)}\ \frac{1}{\log C}\log\frac{R^n[|n\alpha a_n + a_{n-1}| + n\alpha[\alpha_n + \beta_n] + [\alpha_{n-1} + \beta_{n-1}] - \alpha[\alpha_1 + \beta_1] - n[\alpha_0 + \beta_0] + |\alpha a_1 + na_0|]}{|\alpha a_1 + na_0|}, \quad \text{if } R \geq 1$$

and

$$\text{(ii)}\ \frac{1}{\log C}\log\frac{R[|n\alpha a_n + a_{n-1}| + n\alpha[\alpha_n + \beta_n] + [\alpha_{n-1} + \beta_{n-1}] - \alpha[\alpha_1 + \beta_1] - n[\alpha_0 + \beta_0] + |\alpha a_1 + na_0|]}{|\alpha a_1 + na_0|}, \quad \text{if } R \leq 1.$$

Hence the number of zeros of $D_\alpha P(z)$ in $|z| \leq \frac{R}{C}$ $(C > 1, R > 0)$ is equal to the number of zeros of $Q(z)$ in $|z| \leq \frac{R}{C}$ $(C > 1)$.

This completes the proof of Theorem 1.

Proof of Theorem 2. Consider the polar derivative of the polynomial $P(z)$ as in the proof of Theorem 1. From equation (1) in the proof of Theorem 1, we have

$$\begin{aligned}
|Q(z)| \leq\ & |n\alpha a_n + a_{n-1}|R^n \\
& + \sum_{i=0}^{n-2}|[[i+2]\alpha\alpha_{i+2} + ([n-(i+1)] - [i+1]\alpha)a_{i+1} - (n-i)\alpha_i]|R^i \\
& + \sum_{i=0}^{n-2}|[[i+2]\alpha\beta_{i+2} + ([n-(i+1)] - [i+1]\alpha)\beta_{i+1} - (n-i)\beta_i]|R^i \\
& + |\alpha a_1 + na_0| \\
\leq\ & |n\alpha a_n + a_{n-1}|R^n
\end{aligned}$$

$$+\sum_{i=0}^{n-2}[(n-i)\alpha_i+([i+1]\alpha-[n-(i+1)])\alpha_{i+1}-[i+2]\alpha\alpha_{i+2}]R^i$$

$$+\sum_{i=0}^{n-2}[(n-i)\beta_i+([i+1]\alpha-[n-(i+1)])\beta_{i+1}-[i+2]\alpha\beta_{i+2}]R^i$$

$$+|\alpha a_1+na_0|$$

$$\leq\begin{cases}R^n[|n\alpha a_n+a_{n-1}|-n\alpha[\alpha_n+\beta_n]-[\alpha_{n-1}+\beta_{n-1}]+\alpha[\alpha_1+\beta_1]\\ \quad+n[\alpha_0+\beta_0]+|\alpha a_1+na_0|], \quad \text{if } R\geq 1,\\ R[|n\alpha a_n+a_{n-1}|-n\alpha[\alpha_n+\beta_n]-[\alpha_{n-1}+\beta_{n-1}]+\alpha[\alpha_1+\beta_1]\\ \quad+n[\alpha_0+\beta_0]]+|\alpha a_1+na_0|, \quad \text{if } R\leq 1.\end{cases}$$

Applying Lemma 1 to $Q(z)$, we get that the number of zeros of $Q(z)$ in $|z|\leq\frac{R}{C}$ $(C>1,\ R>0)$, does not exceed

(i) $$\frac{1}{\log C}\log\frac{R^n[|n\alpha a_n+a_{n-1}|-n\alpha[\alpha_n+\beta_n]-[\alpha_{n-1}+\beta_{n-1}]+\alpha[\alpha_1+\beta_1]+n[\alpha_0+\beta_0]+|\alpha a_1+na_0|]}{|\alpha a_1+na_0|}, \quad \text{if } R\geq 1$$

and

(ii) $$\frac{1}{\log C}\log\frac{R[|n\alpha a_n+a_{n-1}|-n\alpha[\alpha_n+\beta_n]-[\alpha_{n-1}+\beta_{n-1}]+\alpha[\alpha_1+\beta_1]+n[\alpha_0+\beta_0]+|\alpha a_1+na_0|]}{|\alpha a_1+na_0|}, \quad \text{if } R\leq 1.$$

Hence the number of zeros of $D_\alpha P(z)$ in $|z|\leq\frac{R}{C}$ $(C>1,\ R>0)$ is also equal to the number of zeros of $Q(z)$ in $|z|\leq\frac{R}{C}$ $(C>1)$.

This completes the proof of Theorem 2.

Similarly we can prove Theorem 3 and Theorem 4.

References

[1] G. Eneström, Remarquee sur un théorème relatif aux racines de l'equation $a_n + \cdots + a_0 = 0$ oü tous les coefficient sont et positifs, Tôhoku Math. J. 18 (1920), 34-36.

[2] S. Kakeya, On the limits of the roots of an algebraic equation with positive coefficient, Tôhoku Math. J. 2 (1912-1913), 140-142.

[3] P. Ramulu, Some generalization of Eneström-Kakeya theorem, International Journal of Mathematics and Statistics Invention (IJMSI) 3(2) (2015), 52-59.

[4] P. Ramulu and G. L. Reddy, On the Enestrom-Kakeya theorem, Int. J. Pure Appl. Math. 102(4) (2015), 687-700.

[5] P. Ramulu, G. L. Reddy and C. Gangadhar, On the zeros of polar derivatives of polynomials, Journal of Research in Applied Mathematics 2(4) (2015), 07-10.

[6] Q. G. Mohammad, On the zeros of the polynomials, Amer. Math. Monthly 72(6) (1965), 631-633.

[7] M. H. Gulzar, Number of zeros of a polynomial in a given circle, Research Inventy: International Journal of Engineering and Science 3(10) (2013), 12-17.